

Massive Charged Strings in the Description of Vortex Ring Quantum Nucleation

Uwe R. Fischer*

Institut für Theoretische Astrophysik, University of Tübingen, Germany

We demonstrate the way in which vortex rings in neutral superfluids are equivalent to massive, charged, elastic strings in an electromagnetic field defined locally on the string from a Kalb-Ramond gauge field. We argue that the action thus obtained describes an intermediate scale of vortex motion with phonon fluctuations of the line between the incompressible hydrodynamic régime and the microscopic one dominated by roton emission and absorption. The formalism gives an accurate semiclassical picture of vortex string motion and is of relevance for the description of vortex ring quantum nucleation in a perturbation theory of the purely incompressible case.

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We will discuss here in a condensed form the motion of a slightly deformed vortex ring as a massive, elastic string object under the influence of the nondissipative Magnus force and clarify the requirements for the validity of this formulation.

In the hydrodynamic limit of a 2+1d superfluid there exists a direct correspondence between the genuinely relativistic electron-positron pair interaction via photons in QED and a ‘relativistic’ point-vortex-point-antivortex pair interaction via phonons.^{1,3} To obtain the motion of a vortex line, we extend this analogy to the 3+1d case by introducing elastic energy of the vortex *string*. The ‘relativistic’ result for vortex motion in 2+1d as well as 3+1d will break down long before the local vortex velocity reaches the velocity of sound because the hydrodynamic treatment will cease to be valid. The microscopic excitations and vortex structure will come into play and the limiting velocity of vortex motion will be of the order of the Landau critical

*e-mail: fischer@tat.physik.uni-tuebingen.de

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velocity of roton creation v_L rather than the sound velocity. To describe the failure of the structureless entity approximation of points and strings one usually introduces the ultraviolet cut-off ξ related to the core radius of the vortex and the microscopic excitations of the superfluid, i.e. its many-body structure. Using the notion *string* for a one-dimensional object with an extension ξ in the normal direction to the actual string location with only phonons as excitations of the superfluid amounts to the requirement that at any point on the string the curvature is much less than ξ^{-1} or equivalently that it moves locally much slower than the speed of sound c_s . We consequently use in the Lagrangian the non-‘relativistic’ limit for the motion of the vortex (whereas the phonon background is necessarily ‘relativistic’). The local frequency of vortex motion will then be much slower than a critical $\omega_\xi \equiv c_s/\xi$. Calculating ω_ξ with ξ slightly above half the interparticle spacing in He II (the specific example of superfluid we have in mind here), yields indeed a value of ω_ξ very close to the frequency of the roton minimum.

In what follows, $X^\mu(t, \sigma)$ designates the spacetime location of the string (with σ the arc length parameter), $\dot{X}^\mu \equiv \partial X^\mu / \partial t$, $X'^\mu \equiv \partial X^\mu / \partial \sigma$. The speed of sound $c_s \equiv 1$ in the vortex self-action is understood, except where c_s is explicitly indicated for the convenience of the reader. To generalize the 2+1d result to the 3+1d case we set up a local righthanded basis on the string given by the negative normal, tangent and binormal of the line: $\vec{e}_1(t, \sigma) = -\vec{X}''/|\vec{X}''|$, \vec{X}' , $\vec{e}_2(t, \sigma) = \vec{X}' \times \vec{e}_1$. We first consider the dynamics of the vortex described by the vector \vec{Q} lying in the plane spanned by \vec{e}_1, \vec{e}_2 coming from its local self-interaction. Only in the case of small perturbations from an equilibrium configuration the equations of motion of vortices do obey a Hamiltonian structure.⁶ In particular, self-crossings of a line bending back on itself have to be excluded. For a three-dimensional superfluid such an equilibrium configuration is given by a circular vortex, for which $\vec{X}' = \vec{e}_\Phi$, $\vec{e}_1 = \vec{e}_R$, $\vec{e}_2 = \vec{e}_Z$ and $Q^1 = R$, $Q^2 = Z$. Under this prescription of small perturbations of equilibrium, the vortex self-action is now written as a sum of its static part and a wave-dynamical one, quadratic in derivatives of \vec{Q} :

$$S_{\text{self}}[\vec{Q}(t, \sigma)] = - \int \int dt d\sigma \sqrt{\gamma(t, \sigma)} \left\{ M_0 - \frac{1}{2} M_0 \dot{\vec{Q}}^2 + \frac{1}{2} \frac{\alpha}{\gamma} \vec{Q}'^2 \right\}. \quad (1)$$

Here, $\sqrt{\gamma(t, \sigma)} d\sigma$ is the measure of the string’s proper length (for the circular one $\sqrt{\gamma} = R(t)$). The canonical way to obtain the static hydrodynamic kinetic mass M_0 of the vortex per unit proper length is to calculate an effective action as a functional of the vortex coordinate alone.^{1,2,3,4} The same result, though, is obtained simply by considering the vortex string as a fundamental object in the superfluid. The mass M_0 is then the hydrodynamic vortex energy per unit proper length in the local vortex rest frame (the renormalized

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string tension), divided by the sound speed squared. Reinstating c_s for this Einsteinian mass-energy relation:

$$M_0 c_s^2 = \frac{m \rho_0 \kappa^2}{4\pi} \left[\ln \left(\frac{8R_c}{\xi e^C} \right) \right]. \quad (2)$$

The infrared cut-off in the logarithm is in the static limit equal to the mean distance of line elements, i.e. proportional to the local curvature radius $R_c = |\vec{X}''(t, \sigma)|^{-1} \gg \xi$. The constant C depends on the core structure and has order unity. For ‘relativistic’ vortex motions approaching the frequency ω_ξ the kinetic mass will depend on the position on the string respectively the wavevector of its oscillations.^{3,4} In general, the cut-off related elastic coefficient α (parameterized by the longitudinal as well as the transversal string core structure) is also a function of t, σ . We do not consider short wavelength perturbations, where this dependence becomes significant, and take $\alpha \equiv M_0$. This specific choice in the elastic energy of the string is related to a cut-off in the localized self-induction approximation of classical hydrodynamics.⁷ It corresponds to the assumption that in the long wavelength limit we are using, only massless sound excitations propagating along the string can survive and accounts for the fact that $\alpha \equiv M_0 c_s^2$ must remain finite in the incompressible limit $c_s \rightarrow \infty$.

The linear coupling of the vorticity to the background gauge potential which we will add to (1) represents the Magnus force term in the action.^{8,5} It is a topological coupling, involving no local property of the string, which in its most general, relativistic form reads:

$$S_M = m \rho_0 \kappa \int d^2 \zeta b_{\mu\nu} \dot{X}^\mu X'^\nu. \quad (3)$$

We represent the linear coupling of the background flow to the vorticity by using the dual transformation to the antisymmetric gauge tensor $b_{\mu\nu}$, first expounded in its relation to string interactions and superfluids in now-classic papers.^{9,10} It relates the only dynamical degree of freedom of the order parameter we consider, the phase θ , to $b_{\mu\nu}$ and its field strength $H_{\alpha\mu\nu}$:

$$\frac{\hbar}{m} \partial^\gamma \theta \epsilon_{\gamma\alpha\mu\nu} = v^\gamma \epsilon_{\gamma\alpha\mu\nu} \equiv H_{\alpha\mu\nu} = b_{\mu\nu, \alpha} + b_{\alpha\mu, \nu} + b_{\nu\alpha, \mu}. \quad (4)$$

The self-action (1) with $\lambda = M_0$ could have been obtained from (3) if we included in the Goldstone part of $b_{\mu\nu}$ the contribution of the vortex itself and integrated out the transverse phonon fluctuations of the line up to the relevant infrared and ultraviolet cut-offs. Here, $b_{\mu\nu}$ is solely the background flow gauge potential with wavevector below the cut-off $k_0 \approx e^C/8R_c$ in (2):¹

$$b_{\mu\nu}(x) \equiv \frac{1}{(2\pi)^4} \int_{k < k_0} d^4 k \tilde{b}_{\mu\nu}(k) \exp[ik_\mu x^\mu]. \quad (5)$$

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The Magnus force on a string is equivalent to a local form of the Lorentz force on a particle. This becomes apparent if we define the local ‘electromagnetic’ 3-potential and 3-current density via

$$(m\rho_0)^{-1/2}a_\mu \equiv b_{\mu\sigma} \equiv b_{\mu\nu}X'^\nu, \quad (m\rho_0)^{-1/2}j^\mu \equiv \omega^{\mu\sigma} \equiv \omega^{\mu\nu}X'_\nu, \quad (6)$$

i.e. project the components of $b_{\mu\nu}$ respectively $\omega^{\mu\nu}$ on the local vortex axis, according to (3). In an arrangement of cylindrical symmetry, fulfilled by a ring vortex ($\sigma = \Phi$), the 3-potential is the vector $a^\mu = (a^0, a^r, a^z) = (m\rho_0)^{1/2}(-b_{0\phi}, b_{r\phi}, b_{z\phi})$. The component $b_{0\phi}$ equals for stationary flows Stokes’ stream function.¹¹ We identified $q \equiv (m\rho_0)^{1/2}\kappa$ with the ‘charge’ of the vortex. The ‘electric’ and ‘magnetic’ fields in the Magnus (Lorentz) force law resulting from the linear coupling of the vortex matter coordinate to the local gauge field are represented by an analogous projection

$$F_{\mu\nu} \equiv (m\rho_0)^{1/2}H_{\mu\nu\beta}X'^\beta = (m\rho_0)^{1/2}\epsilon_{\mu\nu\alpha\beta}v^\alpha X'^\beta, \\ \vec{E} = (m\rho_0)^{1/2}\vec{v} \times \vec{X}', \quad \vec{B} = (m\rho_0)^{1/2}v_0\vec{X}', \quad (7)$$

where we used the background flow $v_\mu = (\hbar/m)\partial_\mu\theta$ from the dual transformation (4). In a Galilei invariant condensed matter system like He II, $v^0 = 1$ applies (the velocity of light $c \equiv 1$).

Summing (1) and (3), we obtain the total vortex action:

$$S_V = \int \int dt d\sigma \left\{ -\sqrt{\gamma}M_0 \left(1 - \frac{1}{2}\dot{\vec{Q}}^2 + \frac{1}{2\gamma}\vec{Q}'^2 \right) - qa^0 + q\vec{a} \cdot \dot{\vec{Q}} \right\}, \quad (8)$$

and stress again that it is useful as long as we are able to neglect retardation effects in the motion of the vortex string, i.e. as long as the vortex moves non-‘relativistically’. The fact that only linear and quadratic powers of \dot{Q}_A appear in the Lagrangian for non-‘relativistic’ motions can equivalently be understood as the condition of adiabaticity for the vortex motion:⁴ If the effective action is written as a power series in \dot{Q}_A , the terms in the action (8) are the first two. The canonical vortex momenta for the action (8) are given by the expression

$$\vec{P}(t, \sigma) = \vec{e}_A \frac{\delta L}{\delta \dot{Q}_A(t, \sigma)} = \sqrt{\gamma}M_0\dot{\vec{Q}}(t, \sigma) + q\vec{a}. \quad (9)$$

The Hamiltonian of the vortex is equivalent to that of a continuous string sequence of particles enumerated by σ , bound together by the elastic term $(1/2)M_0\vec{Q}'^2$, which is subject to an ‘electromagnetic’ 3-potential a^μ locally defined at the position of the string from the external gauge field $b_{\mu\nu}$ in (6):

$$H_V[\vec{Q}, \vec{P}] = \int d\sigma \left\{ \sqrt{\gamma} \left[M_0 + \frac{1}{2\gamma M_0} (\vec{P} - q\vec{a})^2 + \frac{M_0}{2\gamma} \vec{Q}'^2 \right] + \frac{1}{2}qa_C^0 + qa_{v_s}^0 \right\}$$

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$$\begin{aligned}
&= \int d\sigma \left[\sqrt{\gamma} M_0 + \frac{1}{2} q a_C^0 + q a_{v_s}^0 \right] \\
&\quad + \frac{1}{2} m \rho_0 \lambda \int d\sigma \sqrt{\gamma} \frac{\bar{\gamma}}{\gamma} \left[\left((\vec{P} - q \vec{a}) / M_0 \right)^2 + \vec{Q}'^2 \right] \equiv H_0 + H_\lambda. \quad (10)
\end{aligned}$$

We used the Coulomb (transverse) gauge condition $\text{div } \vec{a} = 0 \Leftrightarrow b_{\sigma A}{}^A = 0$ and defined $a^0 \equiv a_C^0 + a_{v_s}^0$, i.e. separated off the Coulomb and background velocity parts of the scalar potential. This gives the correct factor of 1/2 in the Coulomb interaction energy of the vortex with other vortices contained in the background part of the energy. In the second part of expression (10) we introduced a scale \bar{R}_c characterizing typical curvature radii of vortices under consideration. We defined $\bar{\gamma} \equiv \gamma(\bar{R}_c)$ and neglected the weak dependence of $\lambda \equiv M_0/m\rho_0\bar{\gamma}$ on σ . For a ring vortex with radius R the quantity $\lambda = (\kappa^2/4\pi R^2) \ln(R/\xi e^C) \approx (\xi/R)^2 \ln(R/\xi e^C)/\pi$, where use was made of the approximate equality $\kappa \approx c_s 2\xi$. Thus the parameter λ gives an approximate measure (that is, up to logarithmic renormalization) of the inverse number of particles $1/N$ contained in a disc of radius \bar{R}_c .

If we wish to describe in the framework of our formalism the nucleation of the string as a quantum object, the canonical commutation relations are to be imposed:

$$\begin{aligned}
[Q_A(\sigma), P_B(\sigma')] &= i\hbar \delta_{AB} \delta(\sigma - \sigma'), \\
[Q_A(\sigma), Q_B(\sigma')] &= [P_A(\sigma), P_B(\sigma')] = 0. \quad (11)
\end{aligned}$$

It should be strongly emphasized that we are dealing here with a vortex string which we have given an effective mass. Were the compressibility zero and hence $M_0 = 0$, we could not impose that canonical coordinates and momenta in the directions perpendicular to the string commute because they are then mutually canonically conjugated up to a factor.^{12,6} If we solve the static background equation of constant ‘magnetic’ field $\vec{B} = \text{rot } \vec{a} = -(m\rho_0)^{1/2} \vec{e}_\Phi$ in the Coulomb gauge, we have the possible gauge choice $a^z = 1/2(m\rho_0)^{1/2} R^2$. This gauge choice is such as to make the integrated expression (9) coincide with the Kelvin momentum of the vortex and prevents the imposition of (11). Any equivalent choice, like the isotropic one $a^z = 1/3(m\rho_0)^{1/2} R^2$, $a^r = -1/3(m\rho_0)^{1/2} RZ$, will do the same.

The Hamiltonian (10) has been split into a part H_0 belonging to a zero mass vortex without elastic energy and a perturbation part H_λ involving the wave term. The Hamiltonian H_0 has been used¹² to compute the quantum nucleation probabilities for vortex half ring formation in the presence of a half sphere at a boundary. The part H_λ represents a small correction and a perturbation theory based on the separation of H_0 and H_λ can be constructed to incorporate the dynamics of the massive, elastic vortex string in the calculation of such nucleation probabilities.

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The microscopic examination of vortex ring quantum nucleation in the context of a U(1) field theory in analogy to pair creation in QED,^{13,14} in which compressibility and vortex mass play a dominant role, suffers from the inadequacy of the field theory in a real superfluid on scales of the coherence length respectively for velocities approaching the speed of sound. In contrast, the formalism developed here gives an accurate idea of what happens dynamically if the microscopic domain in a real superfluid is approached from above. The equations of motion following from the action (8) will be relevant for the description of vortex ring quantum nucleation in an intermediate region between incompressible hydrodynamics and the microscopic domain because they are strictly valid in this semiclassical region. It is thus possible to calculate the corrections the effective mass of a vortex string induces in the intrinsic nucleation probabilities of quantized vortices at the absolute zero of temperature in He II. Our model should be verifiable with the outcome of experiments on intrinsic vortex nucleation in this superfluid,¹⁵ where the plateau of the critical velocity temperature dependence below about 150 mK is usually associated with the onset of a temperature independent quantum régime of nucleation.

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